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Gelfand numbers and widths

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Abstract

In general, the Gelfand widths $\widetilde{c}_n(T)$ of a map T between Banach spaces X and Y are not equivalent to the Gelfand numbers $c_n(T)$ of T. We show that $\widetilde{c}_n(T) = c_n(T)$ $(n \in \mathbb{N})$ provided that X and Y are uniformly convex and uniformly smooth, and T has trivial kernel and dense range.

Keywords: Gelfand numbers; Gelfand widths; s-numbers; Banach spaces; Compact maps

1. Introduction

Widths play an essential role in approximation theory. After their introduction, the theory of s-numbers was developed following their axiomatic introduction by Pietsch (see [7,8]). While linear widths are well-known to be equivalent to the corresponding s-numbers, namely the approximation numbers, and also some other widths are equivalent to their related s-numbers, this is not so for Gelfand widths and numbers. More precisely, let X and Y be Banach spaces and suppose that T is a bounded linear map from X to Y. The Gelfand numbers $c_n(T)$ of T are defined by

$$c_n(T) := \inf\left\{ \left\| T J_M^X \right\| : \operatorname{codim} M < n \right\} \quad (n \in \mathbb{N}),$$

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where J_M^X is the natural embedding from the closed linear subspace M of X into X; and its Gelfand widths $\widetilde{c}_n(T)$ are given by

$$\widetilde{c}_n(T) = \inf_{L_n} \sup\{\|Tx\|_Y : \|x\|_X \leq 1, Tx \in L_n\} \quad (n \in \mathbb{N}),$$

where the infimum is taken over all closed linear subspaces L_n of Y with codimension at most $n-1$. Equivalent definitions of these are

$$c_n(T) = \inf_{x_1^*,\ldots,x_{n-1}^* \in X^*} \left\{\sup\left\{\|Tx\|_Y : x \in B_X, \left\langle x, x_k^*\right\rangle = 0 \text{ for } k < n\right\}\right\} \tag{1}$$

and

$$\widetilde{c}_n(T) = \inf_{y_1^*,\ldots,y_{n-1}^* \in Y^*} \left\{\sup\left\{\|Tx\|_Y : x \in B_X, \left\langle Tx, y_k^*\right\rangle = 0 \text{ for } k < n\right\}\right\}, \tag{2}$$

from which it is clear that $c_n(T) \leq \widetilde{c}_n(T)$. Note that in these alternative descriptions of these quantities it may be supposed that all the x_i^* and y_i^* have unit norm. These quantities provide means of assessing the behaviour of T. The lack of equivalence has not always been recognised in the past; the present authors are among those who have fallen into error on this point (see [3, Chapter 5]; further views concerning these widths and s-numbers may be found in [12, Chapter 1], [10], and in [9, Chapter 6]). The position is clarified by the following example in [2]. As in [9, p. 336], let

$$T_n = I_n \circ Q_n : l_1 \to l_\infty^n \quad (n \in \mathbb{N}),$$

where $Q_n : l_1 \to l_2^n$ is a metric surjection and $I_n : l_2^n \to l_\infty^n$ is the identity map. It is shown that

$$\widetilde{c}_n(T_{2n}) \geq 1/\sqrt{2} \quad \text{and} \quad c_n(T_{2n}) = a_n(T_{2n}) \sim 1/\sqrt{n},$$

where a_n denotes the nth approximation number. Thus $\widetilde{c}_n(T_{2n})/c_n(T_{2n}) \to \infty$ as $n \to \infty$, and $\widetilde{c}_n(T_{2n}) > a_n(T_{2n})$ for all large enough n. Since the approximation numbers are the largest s-numbers, this implies that the Gelfand widths are not s-numbers.

In this paper we show that $\widetilde{c}_n(T) = c_n(T)$ for all $n \in \mathbb{N}$ when X and Y are uniformly convex and uniformly smooth real Banach spaces and T has trivial kernel and range dense in Y. Our primary motivation for establishing this stems from work [4] on the representation of compact maps by means of a series (a Banach space analogue of the celebrated Hilbert space result of Erhard Schmidt) in which this equality plays a crucial role. However, we believe that the result is also of independent interest. Key elements of the proof are the use of James orthogonality [5], and the fact that if two points are close together, then those parts of their polars that lie in the unit ball are close together in the sense of the Hausdorff metric.

2. Preliminaries

Throughout the paper we shall suppose that X and Y are real, uniformly convex and uniformly smooth Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$; the closed unit ball and sphere in X are denoted by B_X and S_X, respectively; T is a bounded linear map from X to Y with trivial kernel, and it is assumed that $T(X)$ is dense in Y. Note that (see [11, Theorems 4.6-C and 4.6-F]) these assumptions on T imply that its adjoint T^* has trivial kernel and range that is dense in X^*. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^*\rangle$, and given any closed linear subspaces M, N of X, X^* respectively, their polar sets are

$$M^0 = \{x^* \in X^* : \left\langle x, x^*\right\rangle = 0 \text{ for all } x \in M\}$$

and

$$^0N = \{x \in X : \langle x, x^* \rangle = 0 \text{ for all } x^* \in N\}.$$

The linear span of a point x will be denoted by sp x.

A map $J_X : X \to X^*$ is defined by the requirement that for all $x \in X$, $J_X(x)$ is the unique norm-attaining functional such that

$$\langle x, J_X(x) \rangle = \|J_X(x)\|_{X^*} \|x\|_X = \|x\|_X^2 .$$

We say that an element $x \in X$ is j-orthogonal (or orthogonal in the sense of James [5]) to $y \in X$, and write $x \perp^j y$, if

$$\|x\|_X \leq \|x + ty\|_X \quad \text{for all } t \in \mathbb{R}.$$

If x is j-orthogonal to every element of a subset W of X, it is said to be j-orthogonal to W, written $x \perp^j W$. A subset W_1 of X is j-orthogonal to $W_2 \subset X$ (written $W_1 \perp^j W_2$) if $x \perp^j y$ for all $x \in W_1$ and all $y \in W_2$.

In general, j-orthogonality is not symmetric, that is, $x \perp^j y$ need not imply $y \perp^j x$.

A decomposition of X in terms of James orthogonality was given by Alber [1], who introduced the following terminology: given closed subsets M_1, M_2 of X, the space X is said to be the James orthogonal direct sum of M_1 and M_2, and we write $X = M_1 \uplus M_2$, if

(1) for each $x \in X$ there is a unique decomposition $x = m_1 + m_2$, where $m_1 \in M_1, m_2 \in M_2$;
(2) $M_2 \perp^j M_1$;
(3) $M_1 \cap M_2 = \{0\}$.

Alber established the following.

Theorem 1. *Let X be uniformly convex and uniformly smooth, and let M be a closed linear subspace of X; let J_X be a duality map that is normalised in the sense that it has gauge function μ with $\mu(t) = t$ for all $t \geq 0$. Then*

$$X = M \uplus J_X^{-1} M^0 \quad \textit{and} \quad X^* = M^0 \uplus J_X M.$$

Finally, given any non-empty, bounded, closed subsets A, B of X, we denote by $\delta(A, B)$ the Hausdorff distance between them:

$$\delta(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}.$$

The function δ is a metric on the space of all such subsets. We shall also need the distance between closed linear subspaces M, N of X defined by

$$d(M, N) = \max \left\{ \sup_{x \in M \cap S_X} \inf_{y \in N} \|x - y\|, \sup_{y \in N \cap S_X} \inf_{x \in M} \|x - y\| \right\}.$$

This is equivalent to

$$\widetilde{d}(M, N) := \delta(M \cap S_X, N \cap S_X);$$

in fact it is easy to see that

$$d(M, N) \leq \widetilde{d}(M, N) \leq \frac{2d(M, N)}{1 + d(M, N)} \leq 2d(M, N). \tag{3}$$

We observe that

$$d(M, N) \leq \delta(M \cap B_X, N \cap B_X) \leq \widetilde{d}(M, N). \tag{4}$$

For

$$\sup_{x \in M \cap S_X} \inf_{y \in N} \|x - y\| = \sup_{x \in M \cap B_X} \inf_{y \in N} \|x - y\| \leq \sup_{x \in M \cap B_X} \inf_{y \in N \cap B_X} \|x - y\|,$$

from which, and the companion inequality with M and N interchanged, the left-hand inequality in (4) follows. Similar considerations give the right-hand inequality. Note also that, by Proposition 1.2 of [6],

$$d(M, N) = d(M^0, N^0). \tag{5}$$

3. The main results

We begin with an immediate consequence of Theorem 1.

Lemma 2. *Let $z^* \in S_{X^*}$ and denote by Z the polar of $\{z^*\}$. Then there exists $z \in S_X$ such that $\langle z, z^* \rangle = 1$ and $z \perp^j Z$. Moreover, each $x \in X$ may be uniquely decomposed as $x = x_1 + x_2$, where $x_1 \in \operatorname{sp} z$, $x_2 \in Z$ and $\|x\| \geq \|x_1\| = \operatorname{dist}(x, Z)$.*

Lemma 3. *Let $\varepsilon > 0$ and suppose that $s^*, z^* \in S_{X^*}$ are such that $\|s^* - z^*\|_{X^*} < \varepsilon/4$; let S, Z be the polars of $\{s^*\}, \{z^*\}$ respectively. Then*

$$\delta(S \cap B_X, Z \cap B_X) < \varepsilon.$$

Proof. Suppose that $\delta(S \cap B_X, Z \cap B_X) \geq \varepsilon$. Then either there exists $x \in S \cap B_X$ such that $\operatorname{dist}(x, Z \cap B_X) > \varepsilon/2$, or there exists $x \in Z \cap B_X$ such that $\operatorname{dist}(x, S \cap B_X) > \varepsilon/2$; without loss of generality suppose the second is the case. By Lemma 2, $X = \operatorname{sp}\{s\} \oplus S$ for some $s \in S_X$, and so $x = x_1 + x_2$ for some $x_1 \in \operatorname{sp}\{s\}$ and $x_2 \in S$, with $\|x\| \geq \|x_1\| = \operatorname{dist}(x, S)$. Thus x_2 is the element of S closest to x. Note that $\|x_1\| \leq 1$ and $\operatorname{dist}(x, S) \leq \operatorname{dist}(x, S \cap B_X)$. If $\|x_2\| \leq 1$, then $x_2 \in S \cap B_X$ and

$$\|x_1\| = \operatorname{dist}(x, S) = \operatorname{dist}(x, S \cap B_X).$$

On the other hand, if $\|x_2\| > 1$, then since $\|x_2\| \leq 1 + \|x_1\|$ and x_2 is the element of S closest to x, there exists $s \in S \cap B_X$ such that $\|s - x_2\| \leq \|x_1\|$. Thus $\|x - s\| \leq \|x_1\| + \|x_2 - s\| \leq 2\|x_1\|$, so that $\operatorname{dist}(x, S \cap B_X) \leq 2\|x_1\|$. It follows that in both cases,

$$\operatorname{dist}(x, S) \leq \operatorname{dist}(x, S \cap B_X) \leq 2 \operatorname{dist}(x, S).$$

Use of Lemma 2 again now shows that

$$\begin{aligned} \langle x, s^* - z^* \rangle &= \langle x, s^* \rangle = \langle x_1, s^* \rangle = \|x_1\| \langle s, s^* \rangle = \|x_1\| = \operatorname{dist}(x, S) \\ &\geq \frac{1}{2} \operatorname{dist}(x, S \cap B_X) > \varepsilon/4. \end{aligned}$$

It follows that $\|s^* - z^*\|_{X^*} > \varepsilon/4$ and we have a contradiction. The lemma follows. □

It is plain from the definitions that $c_n(T) = \widetilde{c}_n(T)$ when $n = 1$. The next lemma shows that this is also true for $n = 2$.

Lemma 4. *The second Gelfand number of T coincides with the second Gelfand width:*

$$c_2(T) = \widetilde{c}_2(T).$$

Proof. Let $\varepsilon > 0$. Given any $z^* \in X^*$, there exists $x_\varepsilon^* \in T^*(Y^*)$ such that $\left\| z^* - x_\varepsilon^* \right\|_{X^*} < \varepsilon$; let Z and X_ε be the polars of $\{z^*\}$ and $\{x_\varepsilon^*\}$ respectively. By Lemma 3,

$$\delta\,(Z \cap B_X, X_\varepsilon \cap B_X) < 2\varepsilon.$$

Hence

$$\begin{aligned}\sup_{x \in Z \cap B_X} \|Tx\| &= \sup\{\|T(x+y-y)\| : x \in Z \cap B_X, x+y \in X_\varepsilon \cap B_X, \|y\| < 4\varepsilon\}\\ &\le \sup\{\|T(x+y)\| + \|Ty\| : x \in Z \cap B_X,\\ &\qquad x+y \in X_\varepsilon \cap B_X, \|y\| < 4\varepsilon\}\\ &\le \sup\{\|T(x+y)\| : x+y \in X_\varepsilon \cap B_X\} + 4\varepsilon \|T\|.\end{aligned}$$

It follows that $\widetilde{c}_2(T) \le c_2(T) + 4\varepsilon \|T\|$, so that $\widetilde{c}_2(T) \le c_2(T)$. As we already know the reverse inequality, the proof is complete. $\square$

Lemma 5. *Let $n \in \mathbb{N} \setminus \{1\}$ and suppose that $s_1^*, \ldots, s_n^*, z^* \in S_{X^*}$, with $s_1^*, \ldots, s_n^*$ linearly independent; let S_i, Z be the polars of $\{s_i^*\}$, $\{z^*\}$ respectively. Then there exists $a > 0$ such that if $\left\| s_n^* - z^* \right\| < \varepsilon$, then*

$$\delta\left(\left(\cap_{i=1}^{n} S_i\right) \cap B_X, \left(\cap_{i=1}^{n-1} S_i\right) \cap Z \cap B_X\right) < a\varepsilon.$$

Proof. By (3),

$$\begin{aligned}\Lambda &:= d\left(\left(\cap_{i=1}^{n-1} S_i\right) \cap S_n, \left(\cap_{i=1}^{n-1} S_i\right) \cap Z\right)\\ &= d\left(\operatorname{sp}\left\{s_1^*, \ldots, s_n^*\right\}, \operatorname{sp}\left\{s_1^*, \ldots, s_{n-1}^*, z^*\right\}\right).\end{aligned}$$

Let

$$A := \left\{(\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{R}, \sum_{i=1}^{n} \alpha_i s_i^* \in B_{X^*}\right\}.$$

Since the s_i^* are linearly independent, they span an n-dimensional subspace S of X^*, and as all norms on a finite-dimensional space are equivalent, $\max_{1 \le i \le n} |\alpha_i|$ is a norm on S equivalent to that induced on it by the norm on X^*: hence

$$b := \max_{(\alpha_1, \ldots, \alpha_n) \in A} |\alpha_n| < \infty.$$

Now let

$$M = {}^0\operatorname{sp}\left\{s_1^*, \ldots, s_n^*\right\}, \qquad N = {}^0\operatorname{sp}\left\{s_1^*, \ldots, s_{n-1}^*, z^*\right\}.$$

Then

$$\Lambda = d\left(M^0, N^0\right) = \max\left(\Lambda_1, \Lambda_2\right),$$

where

$$\Lambda_1 = \sup_{x^* \in M^0 \cap S_X^*} \inf_{y^* \in N^0} \left\| x^* - y^* \right\|$$

and Λ_2 is defined similarly, with M and N interchanged. Hence

$$\begin{aligned}\Lambda_1 &\le \sup\left\{\left\|\sum_{i=1}^{n}\alpha_i s_i^* - \left(\sum_{i=1}^{n-1}\alpha_i s_i^* + \alpha_n z^*\right)\right\| : \sum_{i=1}^{n}\alpha_i s_i^* \in M^0 \cap S_{X^*}\right\}\\ &= \sup\left\{\left\|\alpha_n z^*\right\| : \sum_{i=1}^{n}\alpha_i s_i^* \in M^0 \cap S_{X^*}\right\} \le b\varepsilon.\end{aligned}$$

In the same way it may be shown that $\Lambda_2 \le b\varepsilon$. Thus by (2),

$$\delta\left(\left(\cap_{i=1}^{n-1} S_i\right) \cap S_n \cap B_X, \left(\cap_{i=1}^{n-1} S_i\right) \cap Z \cap B_X\right) \le 2b\varepsilon. \quad \square$$

Corollary 6. *Let $n \in \mathbb{N} \setminus \{1\}$ and for each $i \in \{1, \ldots, n\}$ suppose that $s_i^*, z_i^* \in S_{X^*}$ and let S_i, Z_i be the polars of $\{s_i^*\}, \{z_i^*\}$ respectively; assume that $\{s_1^*, \ldots, s_n^*, z_1^*, \ldots, z_n^*\}$ is linearly independent. There exists $c > 0$ such that if $\left\|s_i^* - z_i^*\right\| < \varepsilon$ for all $i \in \{1, 2, \ldots, n\}$, then*

$$\delta\left(\left(\cap_{i=1}^{n} S_i\right) \cap B_X, \left(\cap_{i=1}^{n} Z_i\right) \cap B_X\right) \le c\varepsilon.$$

Proof. Using the triangle inequality for the Hausdorff metric δ together with Lemma 5, we find that $\delta\left(\left(\cap_{i=1}^{n} S_i\right) \cap B_X, \left(\cap_{i=1}^{n} Z_i\right) \cap B_X\right)$ is bounded above by

$$\begin{aligned}&\delta\left(\left(\cap_{i=1}^{n} S_i\right) \cap B_X, \left(\cap_{i=1}^{n-1} S_i\right) \cap Z_n \cap B_X\right)\\ &\quad + \sum_{k=1}^{n}\delta\left(\left(\cap_{i=1}^{n-k} S_i\right) \cap \left(\cap_{i=n-k+1}^{n} Z_i\right) \cap B_X, \left(\cap_{i=1}^{n-k-1} S_i\right) \cap \left(\cap_{i=n-k}^{n} Z_i\right) \cap B_X\right)\\ &\quad + \delta\left(S_1 \cap \left(\cap_{i=2}^{n} Z_i\right) \cap B_X, \left(\cap_{i=1}^{n} Z_i\right) \cap B_X\right)\\ &\le (n+2)a\varepsilon. \quad \square\end{aligned}$$

After this preparation we are able to establish the main result of the paper.

Theorem 7. *Suppose that X and Y are both uniformly convex and uniformly smooth real Banach spaces, and let $T : X \to Y$ be a bounded linear map with trivial kernel and range dense in Y. Then for all $n \in \mathbb{N}$,*

$$c_n(T) = \widetilde{c}_n(T).$$

Proof. We have simply to deal with the case $n > 2$. Let $\varepsilon > 0$. With the expression (1) for $c_n(T)$ in mind, let $x_1^*, \ldots, x_{n-1}^* \in X^*$; we may suppose that these elements are linearly independent. Since $T^*(Y^*)$ is dense in X^*, there is a set $\{y_i^* : i = 1, \ldots, n-1\} \subset Y^*$ such that, with $z_i^* := T^* y_i^*$ for each i, the set $\{x_1^*, \ldots, x_{n-1}^*, z_1^*, \ldots, z_{n-1}^*\} \subset X^*$ is linearly independent and $\left\|x_i^* - z_i^*\right\|_{X^*} < \varepsilon$ $(i = 1, \ldots, n-1)$. Let X_i, Z_i be the polars of $\{x_i^*\}, \{z_i^*\}$ respectively. Then from (2) we have

$$\widetilde{c}_n(T) \le \sup\left\{\|Tx\|_Y : x \in B_X \cap \left(\cap_{i=1}^{n-1} Z_i\right)\right\}.$$

Put

$$M^{n-1} = \left(\cap_{i=1}^{n-1} X_i\right) \cap B_X, \qquad N^{n-1} = \left(\cap_{i=1}^{n} Z_i\right) \cap B_X.$$

By Corollary 6, $\delta\left(M^{n-1}, N^{n-1}\right) \leq c\varepsilon$. It follows that

$$\sup_{x \in N^{n-1}} \|Tx\| \leq \sup_{x \in M^{n-1}} \|Tx\| + c\varepsilon \|T\|.$$

Thus $\widetilde{c}_n(T) \leq c_n(T)$ and the theorem follows. $\square$

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